A New Algorithm for the Estimation of the Frequency of a Complex Exponential in Additive Gaussian Noise

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***Abstract*—This letter presents a new algorithm for the precise estimation of the frequency of a complex exponential signal in additive, complex, white Gaussian noise. The discrete Fourier transform (DFT)-based algorithm performs a frequency interpolation on the results of an N point complex fast Fourier transform. For large N and large signal to noise ratio, the frequency estimation error variance obtained is 0.063 dB above the Cramer–Rao Bound. The algorithm has low computational complexity and is well suited for real time applications.**

***Index Terms*—Discrete Fourier transform (DFT), fast Fourier transform (FFT), frequency estimation, interpolation.**

I. Introduction

THERE has been substantial prior work in the use of the fast Fourier transform (FFT) to estimate the frequency of a time sampled complex exponential signal in additive white Gaussian noise [1]–[4]. This letter presents a new discrete Fourier transform (DFT)-based method to obtain a very accurate frequency estimate [5]. In particular, the coarse estimate is the frequency corresponding to the maximum amplitude FFT coefficient. Recursion can be done either by frequency translating the signal or frequency modifying DFT coefficients. Frequency modifying the DFT coefficients is suggested in [7]. A frequency error function is defined in terms of two modified DFT coefficients. The first interpolated frequency estimate is the initial frequency estimate plus the output of the frequency error function. Two new modified DFT coefficients are then obtained. The second interpolated frequency is then the first interpolated frequency plus the output of the frequency error function. This iterative computation can be continued until a fixed point solution is obtained in which the input and the output of the frequency error function are zero. Convergence occurs for the third frequency interpolation for the particular frequency error function considered in this letter.

II. Coarse Frequency Estimation

The received signal plus noise, $r[n]$, is given by

$$r[n] = s[n] + \eta[n], \qquad \text{for } n = 0, 1, 2, \ldots, N-1 \tag{1}$$

where $s[n]$ is a complex exponential signal with frequency f given by $s[n] = Ae^{j2\pi f n T_s}$ and $\eta[n]$ is a sequence of independent, identically distributed complex Gaussian random variables with zero mean and variance σ^2. It is desired to process $r[n]$, $n = 0, 1, \ldots, N-1$ to obtain an estimate of f, where f is a fixed and unknown parameter, $\mathrm{f} \in [0, f_s]$. The sampling frequency is $\mathrm{f_s}$, and $T_s = 1/f_s$. The signal-to-noise ratio (SNR) is defined as

$$\mathrm{SNR} = A^2/\sigma^2. \tag{2}$$

Rife and Boorstyn [1] described a technique for the estimation frequency using the FFT. The frequency corresponding to the maximum amplitude FFT coefficient is chosen as a frequency quantized approximation to the maximum-likelihood estimate.

Define

$$\mathbf{r} = \begin{bmatrix} r[0] \\ r[1] \\ . \\ . \\ r[N-1] \end{bmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{bmatrix} Y[0] \\ Y[1] \\ . \\ . \\ Y[N-1] \end{bmatrix}$$

where, $\mathbf{Y} = \mathrm{FFT}(\mathbf{r})$ and $\mathrm{FFT}(.)$ is the N point complex FFT operator. Following Rife and Boorstyn [1], a coarse frequency estimate, $\hat{f}_o$, may be obtained from,

$$k_{\max} = \max{}^{-1}[|Y[k]|\colon 0 \le k \le N-1]$$
$$\hat{f}_o = \frac{k_{\max}}{N} f_s. \tag{3}$$

Assuming that the SNR is sufficiently high, it is highly probable that $f \in [\hat{f}_o - (f_s/2N),\ \hat{f}_o + (f_s/2N)]$. This is an above threshold condition [1]. A fine interpolation may be obtained to improve the frequency accuracy.

III. DFT Interpolation

Define the modified DFT coefficients as α and β as

$$\alpha = Y\left(k_{\max} - \frac{1}{2}\right) = \sum_{n=0}^{N-1} r[n] \exp\left(-j2\pi n \frac{k_{\max} - \frac{1}{2}}{N}\right) \tag{4}$$

$$\beta = Y\left(k_{\max} + \frac{1}{2}\right) = \sum_{n=0}^{N-1} r[n] \exp\left(-j2\pi n \frac{k_{\max} + \frac{1}{2}}{N}\right). \tag{5}$$

A number of possible functional forms, which map α and β to an estimate of the error in $\hat{f}_o$, are possible. The functional mappings contain a particular class [5] characterized by

$$\Delta f_o(\mathbf{r}) = \frac{1}{2N\gamma} \frac{|\beta|^\gamma - |\alpha|^\gamma}{|\beta|^\gamma + |\alpha|^\gamma} f_s, \qquad \text{for } \gamma > 0. \tag{6}$$

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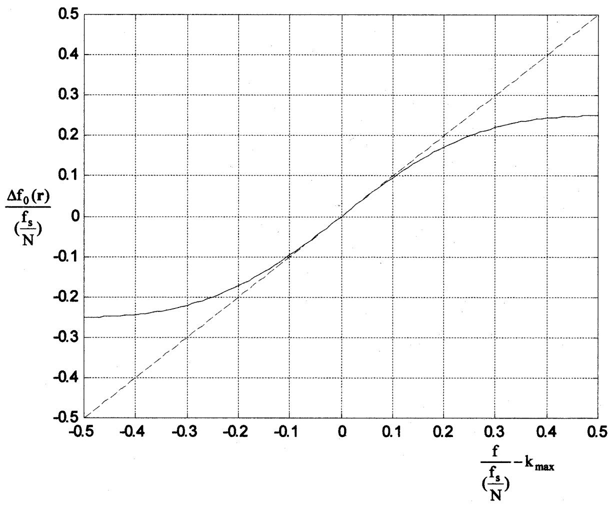

Fig. 1. Normalized frequency discriminant for $\gamma = 2$ as a function of normalized frequency error.

The parameter γ defines the shape of the frequency discriminant function as a function of

$$\left(f - \frac{k_{\max} f_s}{N}\right) \Big/ \frac{f_s}{N}.$$

The application of the frequency discriminant function with $\gamma = 2$ is investigated in this letter. This function is

$$\Delta f_o(\mathbf{r}) = \frac{1}{4N} \frac{|\beta|^2 - |\alpha|^2}{|\beta|^2 + |\alpha|^2} f_s. \tag{7}$$

Fig. 1 shows $\Delta f_o(\mathbf{r})/(f_s/N)$ as a function of $(f/(f_s/N)) - k_{\max}$ for $\gamma = 2$.

From Fig. 1, it is obvious that for $|(f/(f_s/N)) - k_{\max}| < 0.1$, the discriminant yields a good approximation to the actual frequency error. This characteristic may be effectively utilized through the use of recursion.

IV. Recursion Algorithm

The algorithm is defined as

$$\hat{f}_o = \frac{k_{\max}}{N} f_s. \tag{8}$$

For $m = 0, 1, 2, \ldots$, define

$$\alpha_m = \sum_{n=0}^{N-1} r[n] \exp\left(-j2\pi n \left(\frac{\hat{f}_m}{f_s} - \frac{1}{2N}\right)\right) \tag{9}$$

$$\beta_m = \sum_{n=0}^{N-1} r[n] \exp\left(-j2\pi n \left(\frac{\hat{f}_m}{f_s} + \frac{1}{2N}\right)\right) \tag{10}$$

$$\Delta f_m(\mathbf{r}) = \frac{1}{4N} \frac{|\beta_m|^2 - |\alpha_\mathrm{m}|^2}{|\beta_m|^2 + |\alpha_m|^2} f_s \tag{11}$$

$$\hat{f}_{m+1} = \hat{f}_m + \Delta f_m(\mathbf{r}). \tag{12}$$

The interpolated frequency estimate is $\hat{f}_\infty$.

V. Truncation of the Recursion Algorithm

$\hat{f}_\infty$ is the implicit solution of, $\Delta f_\infty(\mathbf{r}) = 0$. $\hat{f}_{m+1}$ is the $m+1$th interpolation. In practice, $\hat{f}_3$ is sufficiently close to $\hat{f}_\infty$ to be used as the interpolated frequency estimate. When $|(f/(f_s/N)) - k_{\max}| > 0.25$, $\hat{f}_1$ will have reasonably large error. However, $|(f - \hat{f}_1)/(f_s/N)| < 0.25$ and therefore $\hat{f}_2$ will have small error and Δf_2 will be close to zero. Therefore, $\hat{f}_3$ will be essentially the implicit solution to $\Delta f_\infty(\mathbf{r}) = 0$. The recursion may be truncated to the evaluation of $\hat{f}_3$.

VI. Reduction of the RMS Error Using Recursion

The rms frequency estimation error due to additive noise, using the proposed frequency discriminant, decreases sharply for $(f - \hat{f})/(f_s/N) \approx 0$. Since the iterative solution adaptively produces a frequency estimate, which is close to this condition, the recursive use of the discriminant produces a frequency estimate with small rms error. By means of the recursion, the discriminant converges to a minimum rms error condition.

VII. Performance of the Algorithm

Using the Taylor series [6], at high signal to noise ratios, the performance of the algorithm has been shown to be [5],

$$\sigma_f^2 = \frac{N \sin^2\left(\frac{\pi}{2N}\right) \tan^2\left(\frac{\pi}{2N}\right)}{4(\mathrm{SNR})\pi^2} \tag{13}$$

where, σ_f^2 is the mean square value of $(\hat{f}_\infty - f)T_s$ and f is a uniform random variable in the interval $[0, f_\mathrm{s}]$.

The Cramer–Rao lower bound (CRLB) on the unbiased frequency estimator mean square error [1] is

$$\sigma_\mathrm{CRLB}^2 = \frac{6}{(2\pi)^2 N(N^2-1)(\mathrm{SNR})}. \tag{14}$$

Therefore, the performance of the frequency estimation algorithm compared to the CRLB is,

$$\frac{\sigma_f^2}{\sigma_\mathrm{CRLB}^2} = \frac{N^2(N^2-1) \sin^2\left(\frac{\pi}{2N}\right) \tan^2\left(\frac{\pi}{2N}\right)}{6}. \tag{15}$$

For large SNR and large N

$$\lim_{N \to \infty} \frac{\sigma_f^2}{\sigma_\mathrm{CRLB}^2} = 10 \log_{10}\left(\frac{\pi^4}{96}\right) = 0.0633 \text{ dB} \tag{16}$$

This limit was also obtained by a different method in [7].

VIII. Simulation of the Performance

The performance of the frequency estimator was obtained by simulation.

Figs. 2 and 3 show the rms estimation error, σ_f, as a function of the SNR in decibels for the cases of a 16-point and a 1024-point FFT, respectively. The results agree with the analysis.

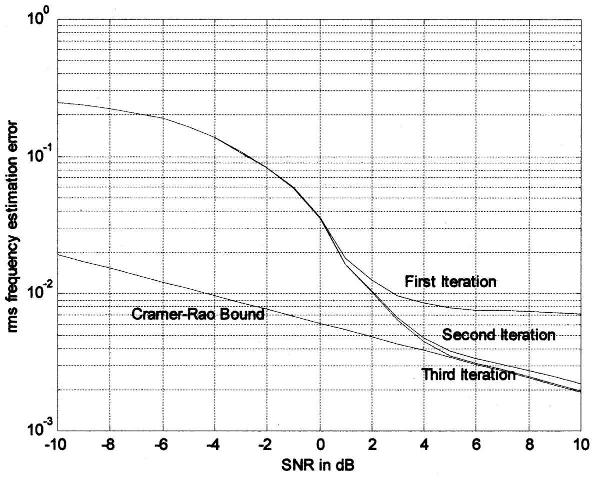

Fig. 2. The rms frequency estimation error as a function of the SNR in decibels. Results are shown of one, two, and three iterations. The results are compared to the CRLB. The FFT length is $N = 16$.

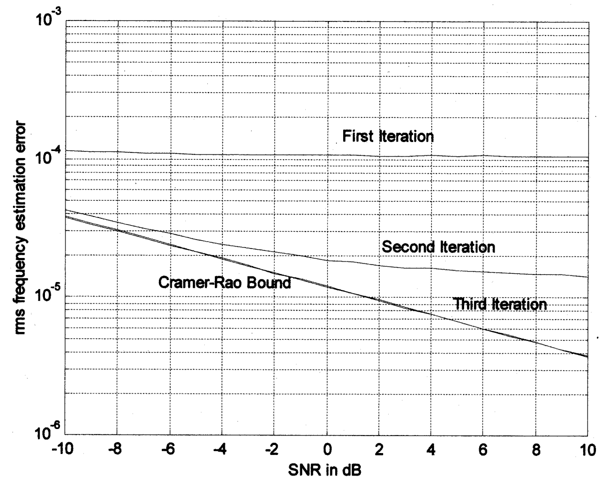

Fig. 3. The rms frequency estimation error as a function of the SNR in decibels. Results are shown for one, two, and three iterations. The results are compared to the CRLB. The FFT length is $N = 1024$.

IX. Conclusion

An algorithm for frequency estimation has been introduced. The high SNR performance of the algorithm is 0.063 dB above the CRLB. The algorithm has low computational complexity and is suitable for real time digital signal processing applications including communications, radar and sonar.

References

[1] D. C. Rife and R. R. Boorstyn, "Single tone parameter estimation from discrete-time observations," *IEEE Trans. Inform. Theory*, vol. IT-20, pp. 591–598, Sept. 1974.

[2] D. C. Rife, "Digital tone parameter estimation in the presence of Gaussian noise," Ph.D. dissertation, Polytechnic Inst. of Brooklyn, New York, NY, June 1973.

[3] B. G. Quinn, "Estimating frequency by interpolation using Fourier coefficients," *IEEE Trans. Signal Processing*, vol. 42, pp. 1264–1268, May 1994.

[4] ——, "Estimating frequency, amplitude and phase from the DFT of a time series," *IEEE Trans. Signal Processing*, vol. 45, pp. 814–817, Mar. 1997.

[5] "Frequency estimation," Australian Provisional Patent 2002950296, Univ. of Technology, Sydney, July 19, 2002.

[6] A. A. Sveshnikov, Ed., *Problems in Probability Theory, Mathematical Statistics and Theory of Random Functions*. New York: Dover, 1968.

[7] E. Aboutanios, "Frequency estimation for low earth orbit satellites," Ph.D. dissertation, Univ. of Technology, Sydney, Australia, 2002.